Linear Algebra: Elementary Transformations of Matrices

In linear algebra, operations known as "elementary transformations" are very important. For example, the following three problems can basically be solved by "repeatedly performing elementary transformations":

Problem 1: Solving a System of Linear Equations
Problem 2: Finding an Inverse Matrix
Problem 3: Finding a Determinant

1.2 How to Perform Elementary Transformations of Matrices
Elementary transformations of matrices are defined as follows.

Definition 1.2.1 (Elementary Transformations of Matrices). The following three transformations of matrices are called elementary row transformations:
(1) Multiplying one row by a ($\neq$0).
(2) Swapping two rows.
(3) Adding a multiplication of the other row by a.

In these three transformations, replacing "rows" with "columns" is called elementary column transformation.

Note 1.2.2. In Problem 1, only elementary row transformations are used.
In Problem 2, only elementary row transformations or We will only use the "column" basic transformation.
Problem 3 uses both basic transformations.

Let's look at some specific examples of the three basic transformations. Let's define matrix A as A = $\begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \end{bmatrix}$

[Transformation of (1): Multiplying the first row by a ($\neq$ 0) gives us

$$\begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2a & 3a & 8a \\ 1 & 2 & 5 \end{bmatrix}$$

Transformation of (2): Swapping the first and second rows gives us

$$\begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix}$$

Transformation of (3): Adding a multiplication of the first row by a to the second row gives us

$$\begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 8 \\ 1+2a & 2+3a & 5+8a \end{bmatrix}$$

The same applies to basic column transformations.

We repeatedly perform these basic transformations to transform the matrix, but without some kind of "goal," it's difficult to know what kind of transformation to perform. Therefore, next, we'll consider the "goal" We define a matrix that satisfies the goal. This is the "reduced matrix" below.

Definition 1.2.3 (Reduced Matrix).
• In a row containing nonzero elements, the first nonzero element counting from the left is called **the principal component** of that row.
• A matrix that satisfies the following conditions (I) to (IV) is called a **reduced matrix**:

(I) A row whose elements are all zero is located below a row containing nonzero elements.
(II) The principal component of a row containing nonzero elements is 1.
(III) The principal component of each row is located further to the right as it descends.
(IV) All components other than the principal component of the column containing the principal component of each row are 0.

Reduced matrices can be complex and difficult to understand in words, but the following example should give you a better idea.

Example 1.2.4 (Reduced Matrix Example).

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, let's introduce an algorithm for transforming a given matrix into a "reduced matrix," which is our "goal." The process of repeatedly performing basic transformations to create a reduced matrix is called "simplification."

Simplification Procedure

Step 1: Of the principal components of each row, Make one of the rows containing the leftmost item the first row using "row elementary transformation **(2)**."

Step 2: Set the principal component of row 1 to 1 using "row elementary transformation **(1)**."

Step 3: Set all components of the column containing the principal component of row 1 other than that principal component to 0 using "row elementary transformation **(3)**."

Step 4: Repeat **Steps 1 to 3** for rows 2 and below, and for rows 3 and below.

Let's actually simplify the previous matrix sample: $A = \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \end{bmatrix}$

The principal component of row 1 is the (1,1) component, **2**, and the principal component of row 2 is the (2,1) component, **1**, so **Step 1** is not necessary.

However, **to simplify** the calculations from **Step 2** onwards, we will simplify rows 1 and 2. Swap the rows.

$$A = \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix} =: \ A_1$$

Since the principal component of row 1 of A_1 is 1, **Step 2** is not necessary. Next, perform **Step 3**. To set the 2 below the (1, 1) component 1 of A_1 to 0, multiply row 2 by (−2) by the value of row.

$$A_1 = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 2+1(-2) & 3+2(-2) & 8+5(-2) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix} =: A_2$$

This completes the operations for row 1. Next, perform **Step 4**, i.e., 2 We move on to the next row. Since there are no rows below the third row,
Step 1 is not necessary. Next, perform **Step 2**. To set the principal component −1 in row 2 to 1, multiply row 2 by (−1).

$$A2 = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 0(-1) & -1(-1) & -2(-1) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix} =: A_3$$

Next, perform **Step 3**. To set the 2 above the (2, 2) component 1 in $A3$ to 0, multiply row 1 by (−2) from row 2.

$$A_3 = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1+0(-2) & 2+1(-2) & 5+2(-2) \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} =: A_4$$

We can confirm that the matrix $A4$ obtained through the above basic transformations is indeed a reduced matrix.

>> End-of-chapter problem: Simplify the following matrix.

(1) $\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -2 & -5 & -1 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 0 & -1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 & -2 \\ -1 & 0 & 1 & 1 & 1 & 3 \\ 2 & 1 & -1 & 0 & -3 & 1 \end{bmatrix}$

Answer:

$$(1)\ \begin{bmatrix}1 & 0 & 5\\ 0 & 1 & -4\end{bmatrix},\quad (2)\ \begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix},\ (3)\ \begin{bmatrix}1 & 0 & -1 & 0 & -2 & 0\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$