The Concept of Differentiation

Differentiation is the rate of change of a function with respect to a small change in a variable. For example, differentiating the function $y = f(x)$ with respect to x, it is how much changes when there is a small change in .

Average Rate of Change/Limit

In the function $y = f(x)$, when the value of x changes from a to b, the change in x is $b - a$, and the change in y is $f(b) - f(a)$. The rate of change (slope) can be expressed as

$$\frac{f(b)-f(a)}{b-a}$$

This is called the "**average rate of change**" when the value of x changes from a to b.

Additionally, for function $f(x)$, when x takes on a different value from a but approaches a value infinitesimally close to it, the value of $f(x)$ approaches a constant value α, which is expressed as

$\lim_{x \to a} f(x) = \alpha$,or $f(x) \to \alpha$ when $x \to a$,

and α is **the limit value** as x approaches a.

The average rate of change of the value of x in a function $f(x)$ as it changes from a to b is called "the derivative of $f(x)$ at $x = a$", and is expressed as $f'(a)$. In the average rate of change of the value of x in a function $f(x)$ as it changes from a to b,

$\frac{f(b)-f(a)}{b-a}$, if we set " $b - a = h$", then the derivative $f'(a)$ of $f(x)$ at " $x = a$" is expressed as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The derivative $f'(a)$ is the slope of the tangent at the point $(a, f(a))$ on the curve "$y = f(x)$".

What is "Differentiation"?

"Differentiation" refers to finding the derivative $f'(x)$ of a function $f(x)$. The derivative $f'(x)$ of a function $f(x)$ is the same as **the differential coefficient** of $f(x)$ at x. The definition of the derivative when differentiating a function with respect to is expressed as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Properties of Derivatives

Let's take a look at some basic properties of derivatives, which are obtained by differentiation.

- If n is a natural number, then、 $(x^n)'$ = nx^{n-1}
- For a constant c, then、 (c)′ = 0
- When $y = k$, then、 $y = k_{f'(x)}$
- When $y = +g(x)$, then 、 y’ = $_{f'(x)}g'(x)$+
- When $y = g(x)$, then、 y’ = $_{f'(x)}$−$g'(x)$
- When $y = k + lg(x)$、 y' = k$_{f'(x)}$+$lg'(x)$

（k and l are constants)）

Increases, Decreases, and Extreme Values of Functions

First, the entire range of values of a real number x that satisfies inequalities such as a≤x≤b or x≥a for real numbers a and b is called an interval.

If $f'(x) > 0$ in a certain interval, the function $f(x)$ increases in that interval. If $f'(x) < 0$ in that interval, the function $f(x)$ decreases in that interval.

Furthermore, when the function $f(x)$ is $f'(a)$ and the sign of $f'(x)$ changes before and after $x = a$, the function $f(x)$ reaches an extreme value at $x = a$.

When $f'(x)$ = 0 and $f(x)$ changes from positive to negative around $x = a$, $f(x)$ is maximum at $x = 0$, and $f(a)$ is the maximum value.

When $f'(a)$ = 0 and $f'(x)$ changes from negative to positive around $x = a$, $f(x)$ is at a minimum at $x = 0$, and $f(a)$ is at a minimum value. These maximum and minimum values together are called extreme values.

The concept of integration

Integration is the inverse operation of differentiation, and integrating a function means finding a function that becomes that function when differentiated.

Indefinite integral

A function $F(x)$ that, when differentiated with respect to a function $f(x)$, becomes $f(x)$ is called a primitive function of $f(x)$. Furthermore, when any constant C is used, $\{F(x)+C\}'$=$f(x)$, so $F(x)+C$ is a primitive function of $f(x)$. Any primitive function of a function $f(x)$, $F(x)+C$, is expressed as ,

$$\int f(x)\,dx$$

and is called the indefinite integral of $f(x)$.

$$\int f(x)\,dx = F(x)+C$$

In this case, the constant C is called the integration constant, and finding the indefinite integral of $f(x)$ is called "integrating" $f(x)$.

When n is a natural number, the derivative is $(x^n)' = nx^{n-1}$, so it can be expressed as...

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

Properties of Indefinite Integrals

Let's introduce some basic properties of indefinite integrals.

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int \{f(x)+g(x)\}dx = \int f(x)dx + \int g(x)dx$$

$$\int kf(x)+lg(x)dx = k\int f(x)dx + l\int g(x)dx$$

Definite Integrals and Area

For a primitive function $F(x)$ of a function $f(x)$ and two real numbers a and b, $F(b) - F(a)$ is called the definite integral of function $f(x)$ from a to b, and can be expressed as follows:

$$\int_a^b f(x)dx$$

In this case, a is called t**he lower end** and b is called **the upper end**. Finding this definite integral is called **integrating the function from** a **to** b.

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

In addition to the properties of definite integrals similar to those of indefinite integrals, definite integrals also have the following properties:

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_c^b f(x)dx + \int_a^c f(x)dx$$

Area Between Two Curves

You can use definite integrals to find the area between two functions.

When $f(x) \geq g(x)$ in the interval $a \leq x \leq b$, the area S of the figure enclosed by two curves $y = f(x)$ and $y = g(x)$ and two lines $x = a$ and $x = b$ can be expressed as...

$$\int_a^b \{f(x) - g(x)\}dx$$

Example Questions to Check Understanding

Here are some example questions on differentiation and integration.

>> Example Question (Differentiation)

① $y = x^3 – 4x^2 + 6x + 24$

② $y = (x – 1)(x + 1)^2$

Answer

①

When $y = f(x) + g(x)$、 $y' = f'(x) + g'(x)$、 then

$y' = (x^3)' - (4x^2)' + (6x)' + (24)' = 3x^2 – 8x + 6$

②

$y = (x – 1)(x + 1)^2 = (x^2 - 1)(x + 1) = x^3 + x^2 – x – 1$, then

$y' = (x^3)' + (x^2)' – (x)' – (1)' = 3x^2 + 2x – 1$

>> Example Problem (Increase/Decrease of Functions, Extreme Values)

Investigate the increase/decrease of the function

$$y = \frac{1}{3}x^3 – 3x^2 + 8x - 3$$

Answer

Differentiating the given function with respect to x,

$y' = x^2 – 6x + 8 = (x - 4)(x - 2)$

If $y' = 0$, then $x = 4, 2$

In this case, the increase/decrease table for y is as follows.

x	⋯	2	⋯	4	⋯
y'	+	0	-	0	+
y'	↗	$\frac{11}{3}$	↘	$\frac{7}{3}$	↗

From the increase/decrease table,

y increases when x≤2, x>4,

and decreases when 2<x≤4.

>> Example Problem (Integration)

Find the following indefinite and definite integrals.

① $\int (3x^2 + 6x - 2)dx$

② $\int_{-1}^{3} (x-1)(x+1)\ dx$

Answer

① $\int \{kf(x) + lg(x)\}dx\ = k\int f(x)\ dx + l\int g(x)\ dx$, then

$$\int (3x^2 + 6x - 2)dx = 3\int x^2 dx + 6\int x dx - 2\int dx$$

$$= 3 \cdot (\frac{1}{3}x^3) + 6 \cdot (\frac{1}{2}x^2) - 2 \cdot x + C$$

$$= x^3 + 3x^2 - 2x + C$$

(C is the integration constant.)

② $\int_{-1}^{3}(x-1)(x+1)dx = \int_{-1}^{3}(x^2-1)dx$

$= [\frac{1}{3}x^3-x]^3 - 1$

$= (9-3)-(-\frac{1}{3}+1)$

$= \frac{16}{3}$

Example Problem (Definite Integrals and Area)

Find the area S of the figure enclosed by the curve $y = x^2-5x+4$ and the x-axis.

Answer

By factorization, x^2-5x+4=(x −1)(x −4). Therefore, the curve $y = x^2-5x+4$ intersects the x-axis at x =1, 4, and since 1≤ x ≤4 and y ≤ 0, the desired area S is as follows:

$$\begin{aligned} S &= -\int_{1}^{4}(x^2-5x+4)dx && (1)\\ &= -[\frac{1}{3}x^3-\frac{5}{2}x^2+4x]_1^4 && (2)\\ &= -(\frac{64}{3}-40+16)+(\frac{1}{3}-\frac{5}{2}+4) && (3)\\ &= \frac{9}{2} && (4) \end{aligned}$$

How to Tackle Calculus

Differential and integral calculus are said to be the most difficult part of high school math, but since differentiation and integration in Mathematics II are fundamental, you'll only learn definitions, concepts, and calculation methods.

The calculations aren't that difficult, so you should be able to solve most problems by simply memorizing the calculation method and formula.

Differential calculus is finding the slope, and if you can draw a table of increase and decrease, you're good to go.

Integration is finding area, so if you can visualize which part of the graph you're looking for, you'll be able to tackle applied problems.

The really difficult part about differentiation and integration is the content of Mathematics III, so don't be too afraid of the "Basics of Differentiation and Integration" in Mathematics II, and just get a grasp of the concepts and definitions.